

Non-integrability of the second Painlevé equation as a Hamiltonian system

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Abstract

We prove non-existence of an additional rational integral for the second Painlevé equation (P_{II}) considered as a Hamiltonian system using Morales - Ramis theory.

Key words: Painlevé second equation, Differential Galois Theory, Hamiltonian system

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1 Introduction

The six Painlevé equations $P_I - P_{VI}$ are non - linear second order differential equations with the property that their solutions have no movable critical points [1]. Except P_I the equations $P_{II} - P_{VI}$ depend on some parameters. These six equations are not reducible to linear equations, nor integrable in terms of previously known functions. Hence they define new transcendental functions for their general solutions called Painlevé transcendents.

Nowadays Painlevé transcendents have important applications in the mathematical physics. We only mention that they arise as reductions of soliton equations such as KdV, nonlinear Schrödinger, Kadomtsev-Petviashvili equation, etc.

Three essential properties of the Painlevé equations are the following [1,2]:

1. They admit a Hamiltonian formulation;
2. They can be expressed as isomonodromic deformation of some linear differential equations with rational coefficients;
3. Except for P_I all the other five equations admit one-parameter families of solutions by means of special functions and also for some special values of the parameters they have particular rational solutions.

As the Painlevé equations are Hamiltonian systems, then their integrability should be considered in the context of Hamiltonian systems. In [2] Morales-Ruiz asked the question to prove rigorously non-integrability of the Hamiltonian systems equivalent to Painlevé equations which have rational particular solutions. In this note we consider P_{II} .

The second Painlevé equation (P_{II})

$$(1) \quad \frac{d^2 w}{dz^2} = 2w^3 + zw + \alpha,$$

where α is a parameter, can be written in the Hamiltonian form $\frac{dq}{dz} = \frac{\partial H}{\partial p}$, $\frac{dp}{dz} = -\frac{\partial H}{\partial q}$ with the Hamiltonian $H(q, p, \alpha) = \frac{1}{2}p^2 - (q^2 + \frac{1}{2}z)p - (\alpha + \frac{1}{2})q$. This system is non-autonomous. Next, we extend it to a two degrees of freedom Hamiltonian system. We

introduce the conjugate variable E to z in the standard way [4], namely putting $H = \frac{1}{2}p^2 - (q^2 + \frac{1}{2}z)p - (\alpha + \frac{1}{2})q - E = 0$. Denote $F := -E$. Then we obtain a Hamiltonian

$$(2) \quad \widehat{H}(q, p, z, F) = \frac{1}{2}p^2 + F - (q^2 + \frac{1}{2}z)p - (\alpha + \frac{1}{2})q$$

defined on $M := \{(q, p, z, F) \in \mathbb{C}^4\}$ with the canonical symplectic structure $dp \wedge dq + dF \wedge dz$. The corresponding autonomous Hamiltonian system reads

$$(3) \quad \begin{aligned} \frac{dq}{ds} &= \frac{\partial \widehat{H}}{\partial p} = p - q^2 - \frac{1}{2}z, & \frac{dz}{ds} &= \frac{\partial \widehat{H}}{\partial F} = 1 \\ \frac{dp}{ds} &= -\frac{\partial \widehat{H}}{\partial q} = 2qp + \alpha + \frac{1}{2}, & \frac{dF}{ds} &= -\frac{\partial \widehat{H}}{\partial z} = \frac{1}{2}p. \end{aligned}$$

The aim of this note is to prove the following theorem using the differential Galois approach.

Theorem 1 *The Hamiltonian system (3), corresponding to P_{II} , does not have second meromorphic first integral for $\alpha = n \in \mathbb{Z}$.*

Remark. We take these values of the parameter $\alpha = n \in \mathbb{Z}$ because it is known that for every $\alpha \in \mathbb{Z}$, P_{II} has unique rational solution. We need simple single-valued solutions to study variational equations along them.

In section 2 we recall some notions and fact about Differential Galois Theory and Morales - Ramis theorem. The proof of Theorem 1 will be given in section 3.

Before ending this section we summarize some facts about P_{II} which are useful for our purpose.

Suppose $\alpha = n \in \mathbb{Z}$ in (1) then

$$(4) \quad w(z, -n) = -w(z, n)$$

which reduces our considerations in Theorem 1 to the cases $n \in \mathbb{N}_0 := 0, 1, 2, \dots$ (see section 3). Next, the Bäcklund transformation

$$(5) \quad w(z, n+1) = -w(z, n) - \frac{2n+1}{2w^2(z, n) + 2w'(z, n) + z},$$

generates the hierarchy of rational solutions of P_{II} from the trivial solution $w(z, 0) = 0$. We list only few of them $w(z, 1) = -\frac{1}{z}$, $w(z, 2) = \frac{1}{z} - \frac{3z^2}{z^3+4}$, $w(z, 3) = \frac{3z^2}{z^3+4} - \frac{6z^5+60z^2}{z^6+20z^3-80}, \dots$

Following [5] we define the Vorobev-Yablonski polynomials $Q_n(z)$ by the recursion relation

$$(6) \quad Q_{n+1} Q_{n-1} = zQ_n^2 + 4(Q'_n)^2 - 4Q_n Q''_n$$

with $Q_0(z) = Q_1(z) = 1$. It turns out that $Q_n(z)$ are monic polynomials of degree $\frac{1}{2}n(n-1)$. Then the rational function

$$(7) \quad w(z, n) = \frac{d}{dz} \left\{ \ln \left[\frac{Q_n(z)}{Q_{n+1}(z)} \right] \right\} = \frac{Q'_n(z)}{Q_n(z)} - \frac{Q'_{n+1}(z)}{Q_{n+1}(z)}$$

satisfies P_{II} . We need also the following two theorems.

Theorem 2 [6] *For every positive integer n , the polynomial $Q_n(z)$ has simple roots.*

Theorem 3 [6] *For every positive integer n , the polynomials $Q_n(z)$ and $Q_{n+1}(z)$ do not have common roots.*

2 Theory

The aim of this section is to formulate basic definitions and facts from Morales - Ramis theory following [3,2]. Let us consider a $2n$ -dimensional complex analytic manifold M and a holomorphic Hamiltonian system X_H on M

$$(8) \quad \frac{d}{dt}z(t) = X_H(z).$$

Let $z = z(t)$ be a non-equilibrium solution of (8). The phase curve Γ is the connected Riemann surface corresponding to $z = z(t)$. The variational equations (VE) along $z = z(t)$ have the form

$$(9) \quad \frac{d}{dt}\xi = X'_H(z(t))\xi, \quad \xi \in T_\Gamma M.$$

Following Ziglin [7] we can always reduce the order of this system by one. Let $N := T_\Gamma M / T\Gamma$ be the normal bundle of Γ and $\pi : T_\Gamma M \rightarrow N$ be the projection. Then the system (9) induces the following system on N

$$(10) \quad \frac{d}{dt}\eta = \pi_*(X'_H(z(t))\pi^{-1}\eta), \quad \eta \in N$$

which is called the normal variational equation (NVE) along Γ . We shall complete the Riemann surface Γ with some equilibrium points and (possibly) points at infinity, in such a way, that the coefficients of the (VE) and of the (NVE) are meromorphic on this extended Riemann surface $\bar{\Gamma} \supset \Gamma$ (see [2,3]).

Morales-Ruiz and Ramis formulate a criterion for non-integrability for Hamiltonian systems in terms of the properties of the differential Galois group G of the normal variational equations [3].

Theorem 4 [2] *Assume there are n first integrals of X_H which are meromorphic in a neighborhood of $\bar{\Gamma}$, in particular meromorphic at infinity and independent in a neighborhood of $\bar{\Gamma}$ (not necessarily on $\bar{\Gamma}$ itself). Then the identity component of the Galois group of the (VE) (resp. (NVE)) is abelian.*

In the applications first a non-equilibrium particular solution is selected. Next, we calculate the VE and NVE. And finally, we have to check if the identity component of differential Galois group G^0 of obtained NVE is Abelian. If it is not, then the Hamiltonian system X_H is not integrable.

Computing the differential Galois group is usually difficult but in the case of second order equation with rational coefficients there exists an efficient algorithm - the Kovacic algorithm. We briefly recall it - see [9,10] for more detailed description.

Any second order differential equation with rational coefficients can be reduced to the form

$$(11) \quad y'' = ry.$$

The logarithmic derivative $\omega := \frac{y'}{y}$ of solution y of (11) satisfies the Riccati equation

$$(12) \quad \omega' + \omega^2 = r$$

and according to Lie - Kolchin's theorem, equation (11) has a Liouvillian solution (that can be expressed via exponentials, integrals and algebraic functions) if and only if the corresponding Riccati equation (12) has an algebraic solution (for definitions, details and proof related to differential algebra see [8,3,9]). Moreover, the degree m of the minimal polynomial for this algebraic solution is one of the following numbers $L_{max} := \{1, 2, 4, 6, 12\}$. The differential Galois group G of (11) is an algebraic subgroup of $SL(2, \mathbb{C})$ and has one of the following forms

1. Case 1: G is triangularisable; in this case equation (11) is reducible and has a solution of the form $y = \exp \int \omega$, where $\omega \in \mathbb{C}(z)$, i.e., Riccati equation (12) has a rational solution ($m = 1$).
2. Case 2: G is "imprimitive" i.e. conjugate to a subgroup of $D \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D$, where D is the diagonal subgroup in $SL(2, \mathbb{C})$; in this case equation (11) has a solution of the form $y = \exp \int \omega$, where ω is algebraic over $\mathbb{C}(z)$ of degree 2, i.e., Riccati equation (12) has an algebraic solution of degree $m = 2$.
3. Case 3: G is finite and cases 1 and 2 do not hold; for this case all solutions of equation (11) are algebraic and Riccati equation (12) has an algebraic solution of degree $m \in \{4, 6, 12\}$.
4. Case 4: $G = SL(2, \mathbb{C})$ and equation (11) has no Liouvillian solution, i.e., Riccati equation (12) has no algebraic solution.

3 Proof of Theorem 1

Consider first the case $\alpha = n \in \mathbb{N}$. The Hamiltonian system (3) possesses a particular solution of the kind

$$(13) \quad q(s) = w(s, n), \quad p(s) = w'(s, n) + w^2(s, n) + \frac{1}{2} z(s),$$

$$z(s) = s, \quad F(s) = \frac{1}{2} \int p \, ds$$

where $w(s, n)$ are already defined rational functions (7). Note that $q(s), p(s), z(s), F(s)$ are rational functions, i.e., the particular solution is single valued. The phase curve Γ of the solution (13) rationally parameterizes the $\widehat{H} = 0$ and naturally extends to $\overline{\Gamma} := \mathbb{CP}^1$.

The variational equations (VE) along $\overline{\Gamma}$ are

$$\frac{d}{ds} \begin{pmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -2w(s, n) & 1 & -\frac{1}{2} & 0 \\ 2p(s) & 2w(s, n) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{pmatrix}$$

and we can take the upper left block as normal variational equations (NVE) (as $z = s$ we substitute $\frac{d}{ds}$ with $\frac{d}{dz}$)

$$(14) \quad \begin{aligned} \frac{d}{dz} \xi_1 &= -2w(z, n) \xi_1 + \eta_1 \\ \frac{d}{dz} \eta_1 &= 2p(z) \xi_1 + 2w(z, n) \eta_1 \end{aligned}$$

In order to study the differential Galois group of (14) we reduce it to a single second order equation

$$(15) \quad \xi'' = (6w^2(z, n) + z) \xi.$$

Now we apply the Kovacic algorithm to the equation (15). Denote

$$r(z) := 6w^2(z, n) + z = 6 \left[\frac{Q_n'^2(z)}{Q_n^2(z)} + \frac{Q_{n+1}'^2(z)}{Q_{n+1}^2(z)} - 2 \frac{Q_n'(z)Q_{n+1}'(z)}{Q_n(z)Q_{n+1}(z)} \right] + z := \frac{R(z)}{S(z)}$$

where $\deg S(z) = \deg Q_n^2(z) Q_{n+1}^2(z) = 2n^2$, $\deg R(z) = \deg z Q_n^2(z) Q_{n+1}^2(z) = 2n^2 + 1$.

Let Y' be the set of zeroes of $S(z)$

$$Y' = \{c \in \mathbb{C} \mid S(c) = 0\} = \{c \in \mathbb{C} \mid Q_n(c) = 0, Q_{n+1}(c) = 0\}$$

and $Y = Y' \cup \{\infty\}$. The order of c denoted by $o(c)$ is the multiplicity of c as a root of $S(z)$ and due to Theorem 2 and Theorem 3, $o(c) = 2$ for every $c \in Y'$. The order of infinity is $o(\infty) = \max(0, 4 + \deg R(z) - \deg S(z)) = 5$.

Define $m^+ := \max_{c \in Y} o(c) = o(\infty) = 5$ and $Y_i = \{c \in Y \mid o(c) = i\}$. In our case

$$Y_2 := \{c \in Y \mid o(c) = 2\} = \{c \in \mathbb{C} \mid S(c) = 0\}, \quad Y_5 := \{\infty\}.$$

Then $\gamma_5 := \text{card } Y_5 = 1$, $\gamma_2 := \text{card } Y_2 = \frac{1}{2}n(n-1) + \frac{1}{2}n(n+1) = n^2$,

so we compute

$$\gamma := \gamma_2 + \sum_{\substack{\text{odd } k \\ 3 \leq k \leq m^+}} \gamma_k = n^2 + \gamma_5 = n^2 + 1 \geq 2.$$

Define $L' \subset L_{\max} : L' = \{2\}$ and since $m^+ = 5 > 2$ then $L \equiv L' = \{2\}$ or $m = 2$, $h(2) = 2$, i.e., for the Galois group of (15) we have only cases 2 or 4.

For every $c \in Y_2$ we need the Laurent series expansion of r around c

$$r = \frac{\alpha_c}{(z-c)^2} + \frac{\beta_c}{z-c} + O(1).$$

Recall that $c = z_j$, $j = 1, \dots, n^2$ are simple roots of $Q_n(z)$ and $Q_{n+1}(z)$. Obviously $\alpha_{z_j} = 6$. It remains to compute $\Delta_{z_j} = \sqrt{1 + 4\alpha_{z_j}} = 5$.

We proceed with the second step of the Kovacic algorithm defining ($m = 2$)

$$\begin{aligned} E_{z_j} &= \{2 - (2 - 2j)\Delta_{z_j} \mid j = 0, 1, 2\} \cap \mathbb{Z} = \{-8, 2, 12\} \quad \text{and} \\ E_\infty &= \{o(\infty)\} = 5. \end{aligned}$$

Finally, in the third step, we note that there is no such $e = \{e_c\}_{c \in Y}$ in the Cartesian product $E := \prod_{c \in Y} E_c$ for which $d(e) = 2 - \frac{1}{2} \sum_{c \in Y} e_c \in \mathbb{N}_0$ since there is only one odd number in the sum - e_∞ and the others are even. Then the Galois group of (15) is $\text{SL}(2, \mathbb{C})$, i.e., not Abelian.

It remains to deal with the case $\alpha = 0$. The particular solution of (3) in this case is

$$(16) \quad q = 0, \quad p = \frac{1}{2} s, \quad z = s, \quad F = \frac{1}{2} \int p ds.$$

In the same manner, the normal variational equation is obtained to be $\xi'' = z\xi$. This is Airy equation and its Galois group is $SL(2, \mathbb{C})$ [9]. Note that from (4) and that $r(z)$ is an even function with respect to n , it follows that above consideration is valid for $\alpha = n \in \mathbb{Z}$. Hence, from Morales-Ramis Theorem (Theorem 4.) the Hamiltonian system is not completely integrable with meromorphic first integrals. This ends the proof.

Remark. Actually, for these α we prove that the system (3) does not possess another rational first integral except \hat{H} —the meromorphic functions on $\bar{\Gamma} = \mathbb{CP}^1$ are exactly the rational ones.

As seen from the proof, equation (15) and Airy equation are not Fuchsian (the point $z = \infty$ is an irregular singular point). Such points are considered the main source of non - integrability because of existence of a non-trivial Stokes phenomenon at them (see [3] for example).

Finally, we recall that in [11] it is shown that P_{II} (with slightly different Hamiltonian) has nonrational (possibly multi - valued) second integral.

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